A FORMULATION OF THE DIFFERENTIAL CONSTITUTIVE EQUATION OF OLDROYD-B FLUID AND WRITING PROGRAMMING CODE IN FREEFEM++ TO SOLVE IT BASED ON FINITE ELEMENT METHOD CONSIDERING TWO AUXILIARY NAVIER-STOKES AND TRANSPORT PROBLEMS

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ABSTRACT

In this paper we study the formulation of the constitutive relation of non-Newtonian visco-elastic fluid flow obeying an Oldroyd-B differential model and decoupling it into two auxiliary problems namely Navier-stokes and transport problem, we write the program in FreeFem++ to solve this model using the finite element method(FEM). We first give the general form of constitutive equations for visco-elastic Oldroyd-B fluid. The unknowns of these equations are $\boldsymbol{\sigma}$ the visco-elastic part of the extra stress tensor, $\mathbf{u}$ the velocity and p the pressure. We solve alternatively a transport equation for the stress and a Navier-Stokes like problem for velocity and pressure. We find the variational formulation of the two auxiliary problems and then we present the programs in FreeFem++ based on finite element method to solve them. We approximate the extra stress, velocity and pressure via P_1 continuous, P_2 continuous and P_1 continuous finite element respectively.

Keywords: Oldroyd-B model, Navier-Stokes equations, transport equation, finite element method, FreeFem++.

INTRODUCTION:

In this paper we present the formulation of Oldroyd-B constitutive equations and decoupling it into two auxiliary problems we develop the program in FreeFem++ to solve them based on finite element method (FEM). The constitutive equations consist of highly non-linear system of partial differential equations of mixed elliptic-hyperbolic form that can be decoupled into a Navier -Stokes system and a tensorial transport equation. The visco-elastic part of extra stress tensor $\boldsymbol{\sigma}$, the velocity $\mathbf{u}$ and the pressure p are the unknowns**.** We assume that the solution $(\boldsymbol{\sigma}, \mathbf{u}, p)$ is sufficiently smooth. We find $(\mathbf{u}, p)$ by solving the Navier-Stokes equations and the tensor $\boldsymbol{\sigma}$ by solving the tensorial transport equation. The approximation of extra stress tensor, velocity and pressure are respectively P_1 continuous, P_2 continuous and P_1 continuous finite element. We begin by describing conservation of laws [6], [14].Then we formulate the constitutive model of Oldroyd-B for incompressible visco-elastic flow. We decouple this constitutive equation into two auxiliary equations namely Navier-stokes equations and tensorial transport equation. We derive the variational formulation of these two auxiliary problems and we introduce finite element approximation of each of the problem [14], [5], [12]. All meshes and simulations are done in FreeFem++ [10]. Using the variational formulation we develop a programming code in FreeFem++ to find $(\mathbf{u}, p)$ from the Navier-Stokes equation and using this $\mathbf{u}$ we will find the extra stress tensor $\boldsymbol{\sigma}$ from the tensorial transport equation using another programming code developed in FreeFem++. We use a problem whose exact solution is known to validate the method. Finally, we address some conclusions and perspective of future works in this field.

CONSERVATION LAWS FOR A CONTINUUM MEDIUM:

Conservation laws states the physical principles governing the fluid motion in a continuum medium. According to the conservation laws, a particular measurable property of an isolated physical system does not change as the system evolves. Lavoisier states that "in nature nothing is created, nothing is lost, everything is transformed". We consider flows of a visco-elastic, incompressible, homogenous fluid in a bounded domain $\Omega \subset IR^d \, (d = 2, 3)$ with boundary $\partial\Omega$. The mathematical formulations of these conservation laws are as follows:

CONSERVATION OF MASS:

Conservation of mass is a fundamental principle of classical mechanics governing the behavior of a continuum medium. It states that in a fixed region, the total time rate of change of mass is identically zero, i.e., mass is neither created nor destroyed during the motion. Physically, this interprets that the rate of change of the density of a fluid in motion is equal to the sum of the fluid convected into and out of the fixed region.
The differential equation expressing conservation of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad (1)$$

where ρ is the density of the fluid, $\mathbf{u}$ is the velocity vector. This equation is also called the equation of continuity.

If the density is a constant, then the flow of the fluid is incompressible and the equation of continuity or the conservation of mass is expressed as

$$\nabla \cdot \mathbf{u} = 0 \qquad (2)$$

CONSERVATION OF MOMENTUM:

The conservation law of momentum for a continuum medium is the extension of the famous Newton's second law of motion, "force= mass× acceleration". For a moving flow field this law describe that the total time rate of change of linear momentum or acceleration of a fluid element is equal to the sum of externally applied forces on a fixed region. The equation of conservation of momentum is given by

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \nabla \cdot \mathbf{T} + \rho \mathbf{f} \qquad (3)$$

where $\mathbf{T}$ is the symmetric tensor field, called Cauchy stress tensor and $\mathbf{f}$ is an external force.

FORMULATION OF THE CONSTITUTIVE RELATIONS:

All materials mostly satisfy the fundamental conservation principles stated above. The mathematical specification of 'material response' laws is said to be the set of constitutive relations. This law relates the

Cauchy stress tensor with the kinematics of different quantities, in particular, the velocity field. Constitutive relations provide us to characterize the mechanical behavior of fluid. In this work we are concerned with non-Newtonian fluids type. It is concerned with the flows of incompressible visco-elastic Oldroyd-B fluids. We first give the general form of constitutive equations and then we give the overview of differential constitutive equations for visco-elastic fluids of Oldryd-B having properties of elastic solids and viscous fluids characterized by a viscous behavior when subject to slow request and elastic behavior subjected to fast request. We take into account several principles and assumptions to formulate a constitutive equation such as principle of determinism, principle of material objectivity. We assume that the stress at a material point is determined by the deformation gradient at this point, i.e. we assume our material is simple fluid.
Under the above principles, for simple, isotropic, incompressible fluid, the Cauchy stress tensor T can be expressed as

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}_s$$

where p is the hydrostatic pressure, $\boldsymbol{\tau}_s$ is the extra stress tensor and **I** is the identity matrix or Kronecker tensor**.**

NON-NEWTONIAN FLUIDS:

MODELS OF NON-NEWTONIAN VISCO-ELASTIC FLUIDS OF OLDROYD TYPE:

Oldroyd observe that the convected time derivative $\frac{d\boldsymbol{\pi}}{dt} = \frac{\partial \boldsymbol{\pi}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\pi}$ of a tensor $\boldsymbol{\pi}$ is not objective. The objective form of the time derivative of a tensor can be expressed as

$$\frac{D_a\boldsymbol{\pi}}{Dt} = \frac{d\boldsymbol{\pi}}{dt} = \frac{\partial \boldsymbol{\pi}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\pi} + \boldsymbol{\pi}\mathbf{W}(\mathbf{u}) - \mathbf{W}(\mathbf{u})\boldsymbol{\pi} - a\left(\boldsymbol{\pi}\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\boldsymbol{\pi}\right) \quad (4)$$

where $-1 \le a \le 1$ is a parameter.

Here $\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right]$ is the rate of deformation tensor or strain rate tensor, $\mathbf{W}(\mathbf{u}) = \frac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^t\right]$ is the rate of spin or vorticity tensor and $\nabla\mathbf{u}$ is the velocity gradient tensor and $(\nabla\mathbf{u})^t$ is the transpose of $\nabla\mathbf{u}$.
Oldroyd suggested a general form of constitutive equation as [11]

$$\lambda_1 \frac{D_a\boldsymbol{\tau}_s}{Dt} + \boldsymbol{\tau}_s + \boldsymbol{\gamma}(\boldsymbol{\tau}_s, \nabla\mathbf{u}) = 2\mu\left[\lambda_2 \frac{D_a\mathbf{D}(\mathbf{u})}{Dt} + \mathbf{D}(\mathbf{u})\right], \quad 0 \le \lambda_2 \le \lambda_1 \quad (5)$$

Where the tensor $\boldsymbol{\tau}_s$ is the extra stress, μ the viscosity coefficient of the fluid which is assumed to be constant, $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$ are the constants depends on the continuous medium are respectively called the relaxation and retardation time of fluid. There are several types of general model. We can generalize these models. For example the extra stress $\boldsymbol{\tau}_s$ can be written as a sum of partial stresses and for each partial stress there is a constitutive equation with different relaxation time.

MODELS OF OLDROYD-B FLUIDS WITH EQUATIONS OF MOTION:

If in **(5),** $\boldsymbol{\gamma}(\boldsymbol{\tau}_s, \nabla\mathbf{u}) = 0$, $\lambda_1 > \lambda_2 > 0$ and $a = 1$ then we say that the constitutive equation is Oldroyd-B type. We know that the Cauchy stress tensor is given by $\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}_s$. Decomposing the extra stress tensor $\boldsymbol{\tau}_s$ into the sum of its Newtonian part $\boldsymbol{\sigma}_n$ and its visco-elastic part $\boldsymbol{\sigma}_e$ we can write $\boldsymbol{\tau}_s = \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e$, where $\boldsymbol{\sigma}_n = 2\mu_n\mathbf{D}(\mathbf{u})$ and $\mu_n = \mu\frac{\lambda_2}{\lambda_1}$. So, $\boldsymbol{\sigma}_n = 2\mu\frac{\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{u})$.

The Cauchy stress tensor can be written as

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e = -p\mathbf{I} + 2\mu\frac{\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{u}) + \boldsymbol{\sigma}_e$$

Considering the above properties, from (5), for Oldroyd-B fluid, the constitutive equation can be written as

$$\lambda_1 \frac{D_a\boldsymbol{\tau}_s}{Dt} + \boldsymbol{\tau}_s = 2\mu\left[\lambda_2 \frac{D_a\mathbf{D}(\mathbf{u})}{Dt} + \mathbf{D}(\mathbf{u})\right]$$

$$\Rightarrow \lambda_1 \frac{D_a(\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e)}{Dt} + \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_e = 2\mu\left[\lambda_2 \frac{D_a\mathbf{D}(\mathbf{u})}{Dt} + \mathbf{D}(\mathbf{u})\right]$$

$$\Rightarrow \lambda_1 \frac{D_a\boldsymbol{\sigma}_e}{Dt} + \lambda_1 2\mu \frac{\lambda_2}{\lambda_1}\frac{D_a\mathbf{D}(\mathbf{u})}{Dt} + 2\mu\frac{\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{u}) + \boldsymbol{\sigma}_e = 2\mu\left[\lambda_2 \frac{D_a\mathbf{D}(\mathbf{u})}{Dt} + \mathbf{D}(\mathbf{u})\right]$$

$$\Rightarrow \lambda_1 \frac{D_a\boldsymbol{\sigma}_e}{Dt} + \boldsymbol{\sigma}_e = 2\mu\left(1 - \frac{\lambda_2}{\lambda_1}\right)\mathbf{D}(\mathbf{u})$$

$$= 2(\mu - \mu_n)\mathbf{D}(\mathbf{u})$$

So, $\lambda_1 \frac{D_a\boldsymbol{\sigma}_e}{Dt} + \boldsymbol{\sigma}_e = 2\mu_e\mathbf{D}(\mathbf{u})$

where $\mu_e = \mu - \mu_n$ i.e., $\mu = \mu_e + \mu_n$.

Using (4) we can finally write,

$$\lambda_1\left(\frac{\partial\boldsymbol{\sigma}_e}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}_e\right) + \boldsymbol{\sigma}_e = 2\mu_e\mathbf{D}(\mathbf{u}) + \mathbf{W}(\mathbf{u})\boldsymbol{\sigma}_e - \boldsymbol{\sigma}_e\mathbf{W}(\mathbf{u}) + \boldsymbol{\sigma}_e\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\boldsymbol{\sigma}_e \qquad (6)$$

From the conservation law of momentum (3), we can write,

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\mathbf{T} + \rho\mathbf{f}$$

$$\Rightarrow \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot[-p\mathbf{I} + 2\mu_n\mathbf{D}(\mathbf{u}) + \boldsymbol{\sigma}_e] + \rho\mathbf{f}$$

After simplifying we get

$$\Rightarrow \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + 2\mu_n\nabla\cdot\mathbf{D}(\mathbf{u}) + \nabla\cdot\boldsymbol{\sigma}_e + \rho\mathbf{f}$$

If $\nabla\cdot\mathbf{u} = 0$, then $2\nabla\cdot\mathbf{D}(\mathbf{u}) = 2\nabla\cdot\frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right] = \Delta\mathbf{u}$.

So, the conservation of momentum can be written as

$$\Rightarrow \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \mu_n\Delta\mathbf{u} = \nabla\cdot\boldsymbol{\sigma}_e + \rho\mathbf{f} \qquad (7)$$

For the simplicity we write $\boldsymbol{\sigma}$ instead of $\boldsymbol{\sigma}_e$. Imposing some boundary conditions we obtain the following Oldroyd-B problems formed by the system of equations (7), (2) and (6):

Find $(\mathbf{u}, p, \boldsymbol{\sigma})$ defined in Ω such that

$$\begin{cases} \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \mu_n\Delta\mathbf{u} = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \lambda_1\left(\frac{\partial\boldsymbol{\sigma}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\sigma}\right) + \boldsymbol{\sigma} = 2\mu_e\mathbf{D}(\mathbf{u}) + \mathbf{W}(\mathbf{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}(\mathbf{u}) + \boldsymbol{\sigma}\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\boldsymbol{\sigma} \quad \text{in } \Omega \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega. \end{cases} \qquad (8)$$

We observe that the conservation of momentum leads to symmetry properties of the tensor $\boldsymbol{\sigma}$, i.e. $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}$.

The problem (8) is a mixed problem. The first two equations form an elliptic system for $(\mathbf{u}, p)$ which is in the form of Navier-Stokes equations. The last equation has a hyperbolic characteristic for each component of $\boldsymbol{\sigma}$ $(\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2})$ which is the type of Transport equation. We will find the solution of (8) in case of steady flow i.e. for $\frac{\partial\mathbf{u}}{\partial t} = 0$ by using two auxiliary problems.

THE NAVIER-STOKES EQUATIONS:

When the visco-elastic part $\boldsymbol{\sigma}_e = 0$, then the Cauchy stress tensor can be written as $\mathbf{T} = -p\mathbf{I} + \boldsymbol{\sigma}_n = -p\mathbf{I} + 2\mu_n\mathbf{D}(\mathbf{u})$.

Substituting the value of $\mathbf{T}$ in (3) and after simplifying, the momentum equation can be written as

$$\Rightarrow \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \mu_n\Delta\mathbf{u} = \rho\mathbf{f}$$

Considering ρ as a constant, we define the kinematic viscosity $\nu = \frac{\mu}{\rho}$ (m^2/s) and the scaled pressure $\frac{p}{\rho}$ (m^2/s^2) still denoted by p and we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f} \tag{9}$$

Imposing homogeneous Dirichlet boundary conditions, the Navier-stokes equations form by (3) and (9) is as follows

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} = 0 \quad \text{on} \ \partial\Omega. \end{cases} \tag{10}$$

If in second equation of (8), the visco-elasticity $\boldsymbol{\sigma}$ is known, then considering $\frac{1}{\rho}(\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f})$ still denoted by $\mathbf{f}$ we get $\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f}$. So the first two equations in (8) become identical with the Navier-Stokes equations. So it is enough to find the solution of Navier-stokes equations.

For steady flow the Navier-stokes problem can be written as

Find $(\mathbf{u}, p)$ such that

$$\begin{cases} \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} = 0 \quad \text{on} \ \partial\Omega. \end{cases} \tag{11}$$

We use some notations of different function spaces details of which can be found in [1], [3]. The variational or weak formulation of Navier-stokes equation consists of the integral equations over Ω obtained by integration, after multiplying the momentum equation and continuity equation by appropriate test functions. Applying the Green's formula for the integration by parts and taking into account that $\mathbf{v}$ vanishes on the boundary and after simplifying we get the variational formulation of the Navier-Stokes problem (11) as follows:

Find $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega)\times\mathbf{L}_0^2(\Omega)$ such that

$$\begin{cases} \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u}).\mathbf{v} - \int_\Omega p\nabla.\mathbf{v} + \nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \mathbf{f}.\mathbf{v} \\ \int_\Omega q\nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}\,|_{\partial\Omega} = 0, \end{cases} \tag{12}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega)\times\mathbf{L}_0^2(\Omega)$.

It can be proved [8] that the problem (12) is well-posed and equivalent to (11). The existence and uniqueness of theorem for the solutions of Navier-Stokes system can be found in [7], [8], [17].

We use classical Galerkin method to find the solution. We write the discretized problem. Let h denotes a discretization parameter and let $\mathbf{V}_h$ and $\mathbf{Q}_h$ be two finite dimensional spaces such that $\mathbf{V}_h \in \mathbf{H}^1(\Omega)$ and $\mathbf{Q}_h \in \mathbf{L}^2(\Omega)$. We let $\mathbf{V}_h^0 := \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ and $\mathbf{M}_h := \mathbf{Q}_h \cap \mathbf{L}_0^2(\Omega)$.

Considering the homogeneous case $\mathbf{u}_h = 0$, in these spaces the discrete finite element approximation problem of (12) can be written as follows:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times \mathbf{M}_h$ such that

$$\begin{cases} \int_\Omega (\mathbf{u}_h\cdot\nabla\mathbf{u}_h).\mathbf{v}_h - \int_\Omega p_h\nabla.\mathbf{v}_h + \nu\int_\Omega \nabla\mathbf{u}_h : \nabla\mathbf{v}_h = \int_\Omega \mathbf{f}.\mathbf{v}_h \qquad \forall\mathbf{v}_h \in \mathbf{V}_h^0 \\ \int_\Omega q_h\nabla\cdot\mathbf{u}_h = 0 \qquad \forall q_h \in \mathbf{M}_h \end{cases} \tag{13}$$

Let N and M be the dimensions of the spaces $\mathbf{V}_h$ and $\mathbf{Q}_h$, and let $\{\varphi_n\}_{n=1,\cdots,N}$ and $\{\psi_m\}_{m=1,\cdots,M}$ be their respective Lagrange bases. We write $\mathbf{u}_h = \sum_{n=1}^{N} \mathbf{u}_n \varphi_n$ and $p_h = \sum_{m=1}^{M} p_m \psi_m$. Setting $\mathbf{v}_h = (v_h^1, v_h^2) = (\{\varphi_r\},0)$ and $(0,\{\varphi_r\})$ for $r = 1, 2, \cdots, N$ and also the test basis for q_h and substituting in problem (13) and integrating, we obtain a nonlinear algebraic system. Details can be found in [5].

The solution of this system can be evaluated using iterative method. All meshes and simulations were done in FreeFem++ [10] which is a free software with its own high level programming language based on the finite element method (FEM) to solve partial differential equations. An automatic mesh generator is used in FreeFem++ based on Delaunay-Voronoi algorithm where the number of internal points are proportional to the number of points on the boundaries. The graphics were generated in FreeFem++.

We develop the program in FreeFem++ from the varaitional problem (12). We use Newton-Raphson iteration [4] for the system of non-linear equations and Frechet derivative is used for the non-linear term. The calculations have been performed with a kinematic viscosity $\nu = 1$. For the validation of the code we consider a model problem whose exact solution is known and is given by

$$\mathbf{u}(\mathbf{x}) = \left((x^2 - x)^2 (y^2 - y)(2y-1), -(x^2 - x)(y^2 - y)^2 (2x-1)\right) \qquad (14)$$

$$p(\mathbf{x}) = x + y \qquad (15)$$

This problem is solved on the unit square $\Omega = [0,1]\times[0,1]$ and prescribes the exact velocity according to (14) and (15) along the boundary of the fluid domain. Since the kinematic viscosity is known, so it results the external force for this specific problem. If the external force is known, then using the following code we can find the velocity and pressure. Here the functions f1 and f2 are two components of external force.

```
int n=50, m=50;            real x0=0.0, x1=1.0;            real y0=0.0, y1=1.0;
mesh Th=square(n,m,[x0+(x1-x0)*x,y0+(y1-y0)*y]); plot(Th); fespace Vh(Th,P2);        fespace Qh(Th,P1);
Vh u1,u2,v1,v2,u1p,u2p; //u1p,u2p means the previous iteration value
Qh p,q,p0;          real nu=1.,error;                        real eps=10e-19;
func     f1=-2*(2*x-1)^2*(y^2-y)*(2*y-1)-(4*(x^2-x))*(y^2-y)*(2*y-1)-6*(x^2-x)^2*(2*y-1)+1+2*(x^2-x)^3*(y^2-
y)^2*(2*y-1)^2*(2*x-1)-(x^2-x)*(y^2-y)^2*(2*x-1)*((x^2-x)^2*(2*y-1)^2+2*(x^2-x)^2*(y^2-y));
funcf2=(6*(2*x-1))*(y^2-y)^2+(2*(x^2-x))*(2*y-1)^2*(2*x-1)+(4*(x^2-x))*(y^2-y)*(2*x-1)+1+(x^2-x)^2*(y^2-
y)*(2*y-1)*(-(2*x-1)^2*(y^2-y)^2-(2*(x^2-x))*(y^2-y)^2)+2*(x^2-x)^2*(y^2-y)^3*(2*x 1)^2*(2*y-1);
problem      SNS([u1,u2,p],[v1,v2,q])       =int2d(Th)(nu*(      dx(u1)*dx(v1)+dy(u1)*dy(v1)+      dx(u2)*dx(v2)
+dy(u2)*dy(v2))-p*q*(0.0000001)-p*(dx(v1)+dy(v2))  - q*(dx(u1)+dy(u2))
+ (u1*dx(u1p)+u2*dy(u1p))*v1 + (u1*dx(u2p)+u2*dy(u2p))*v2
+ (u1p*dx(u1)+u2p*dy(u1))*v1 + (u1p*dx(u2)+u2p*dy(u2))*v2)
-int2d(Th)( nu * ( dx(u1p)*dx(v1) + dy(u1p)*dy(v1)
+ dx(u2p)*dx(v2) + dy(u2p)*dy(v2)) - p0*(dx(v1)+dy(v2))-        q*(dx(u1p)+dy(u2p)))-
int2d(Th)((u1p*dx(u1p)+u2p*dy(u1p))*v1+(u1p*dx(u2p)+u2p*dy(u2p))*          v2)-int2d(Th)(f1*v1+f2*v2)+
on(1,2,3,4,u1=0,u2=0);            u1p = 0;              u2p = 0;
p0=0;          for(int i=0;i<=50;i++) {          SNS;          u1p[]-=u1[];     u2p[]-=u2[];        p0[]-=p[];
error= u1[].linfty + u2[].linfty + p[].linfty;          if (error < eps) break;
          cout<<i<<endl;          cout<<error<<endl;          }
          plot(u1p,value=true,nbiso=20,ps="u1.eps");          plot(u2p,value=true);
          plot(p0,value=true);          plot(coef=1,[u1p,u2p]);
```

Code1: For Navier-Stokes problem

NUMERICAL RESULTS:

We performed the code to find the velocity and pressure defined on mesh of dimension 50×50. The tests that follow were performed with $\nu = 1$. We denote the first component of the velocity field by *u1*, the second component of velocity field by *u2* and the pressure filed by *p*.

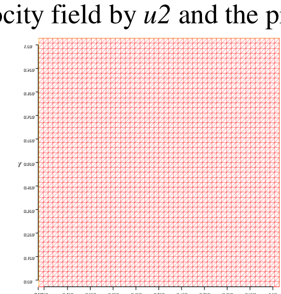

Figure1: 50×50 **mesh**

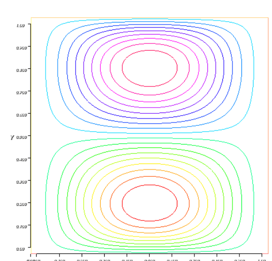

Figure 2: Contours of *u1*

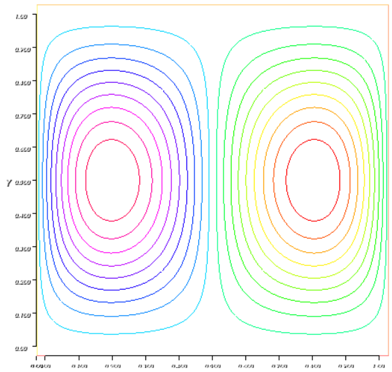

Figure 3: Contours of *u2*

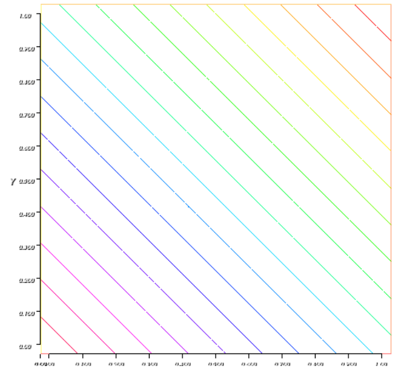

Figure 4: Contours of *p*

From the above figures we observe that we have the contour plot of the velocity and pressure using the which are in fact the plot of the solutions of the Navier-Stokes problem. Numerical data are automatically generated to generate the contour plot. We can easily compare these solutions plot with the exact solutions. For any other problems, if the external force is known, we can easily find their velocity and pressure by changing the components f1 and f2. So using FreeFEm++ the solutions have been found for Navier-Stokes problem.

TRANSPORT EQUATION:

The transport equation is considered as an auxiliary problem to the Oldroyd-B model.

Let $\mathbf{u} \in \mathbf{W}^{1,\infty}(\Omega)$. The tensorial steady transport problem read as

Find $\boldsymbol{\sigma} \in \mathbf{W}^{1,\infty}(\Omega)$ such that

$$\lambda(\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} + \boldsymbol{\sigma} = \mathbf{g} \quad \text{in } \Omega \tag{16}$$

where $\lambda \in IR$ and $\mathbf{g} \in \mathbf{L}^2(\Omega)$.

If we let $\mathbf{g} = 2\mu_e \mathbf{D}(\mathbf{u}) + \mathbf{W}(\mathbf{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}(\mathbf{u}) + \boldsymbol{\sigma}\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\boldsymbol{\sigma}$ and $\lambda = \lambda_1$ then the third equation of (8) is identical with (16) in steady case i.e. when $\frac{\partial \boldsymbol{\sigma}}{\partial t} = 0$. So we solve the transport problem to get the visco-elastic tensor of Oldroyd-B model.

Let us define the space of tensorial functions

$\mathbf{S} = \left\{ \boldsymbol{\sigma} = \{\sigma^{i.j}\}_{i,j=1,2} : \sigma^{1.2} = \sigma^{2.1}, \sigma^{i.j} \in L^2(\Omega), \text{ and } \mathbf{u}.\nabla\boldsymbol{\sigma} \in \mathbf{L}^2(\Omega) \right\}.$

Multiplying the equation (16) by a test tensor $\boldsymbol{\tau} \in \mathbf{S}$ and integrating over Ω and writing as bilinear and linear form, we obtain the variational formulation of (16) as follows:

Find $\boldsymbol{\sigma} \in \mathbf{S}$ such that

$$\lambda \int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\sigma}) : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} = \int_\Omega \mathbf{g} : \boldsymbol{\tau} \quad \text{in} \quad \Omega \tag{17}$$

The existence and uniqueness theorem of weak solutions to this problem can be found in [13], [2].

In the finite dimensional subspace $\mathbf{S}_h \subset \mathbf{S}$, where *h* is a discretization parameter, the discrete tensorial problem is defined as follows:

Find $\boldsymbol{\sigma}_h \in \mathbf{S}_h$ such that

$$\lambda \int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\sigma}_h) : \boldsymbol{\tau}_h + \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau}_h = \int_\Omega \mathbf{g} : \boldsymbol{\tau}_h \quad \forall \boldsymbol{\tau}_h \in \mathbf{S}_h \tag{18}$$

Writing each component of $\boldsymbol{\tau}_h$ we find an algebraic system of linear equation from (18).

For the code in FreeFem++ we use the variational formulation (17). We write $\mathbf{g} = 2\mu_e \mathbf{D}(\mathbf{u}) + \mathbf{W}(\mathbf{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}(\mathbf{u}) + \boldsymbol{\sigma}\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\boldsymbol{\sigma}$ and $\lambda = \lambda_1$. If we write $\mathbf{u}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ component wise such as $\mathbf{u} = (u_1, u_2)$, $\boldsymbol{\sigma} = [\sigma_{ij}]_{i,j=1,2}$ and $\boldsymbol{\tau} = [\tau_{ij}]_{i,j=1,2}$ then we find from (17)

$$\lambda_1\left[\int_\Omega \sum_{i,j=1}^{2}\left(u_1\frac{\partial\sigma_{ij}}{\partial x_1}+u_2\frac{\partial\sigma_{ij}}{\partial x_2}\right)\tau_{ij}+\int_\Omega \sum_{i,j=1}^{2}\sigma_{ij}\tau_{ij}\right]$$
$$=2\mu_e\int_\Omega \sum_{i,j=1}^{2}D_{ij}\tau_{ij}+\int_\Omega \sum_{i,j=1}^{2}\left(\sum_{k=1}^{2}(W_{ik}\sigma_{kj}-\sigma_{ik}W_{kj}+\sigma_{ik}D_{kj}+D_{ik}\sigma_{kj}\right)\tau_{ij} \quad (19)$$

Setting $(\boldsymbol{\tau}=[\tau_{ij}]_{i,j=1,2}$ where $\tau_{ij}=0$ for $i,j\neq 1)$, $(\boldsymbol{\tau}=[\tau_{ij}]_{i,j=1,2}$ where $\tau_{ij}=0$ for $i\neq 1, j\neq 2)$ and $(\boldsymbol{\tau}=[\tau_{ij}]_{i,j=1,2}$ where $\tau_{ij}=0$ for $i,j\neq 2)$

We have the following system of three linear equations in the varaiational formulation of the transport problem:

$$\begin{cases}\lambda_1\int_\Omega\left(u_1\frac{\partial\sigma_{11}}{\partial x_1}+u_2\frac{\partial\sigma_{11}}{\partial x_2}\right)\tau_{11}+\int_\Omega\sigma_{11}\tau_{11}=2\mu_e\int_\Omega\frac{\partial u_1}{\partial x_1}\tau_{11}+2\lambda_1\int_\Omega\left(\frac{\partial u_1}{\partial x_1}\sigma_{11}+\frac{\partial u_1}{\partial x_2}\sigma_{12}\right)\tau_{11},\\ \lambda_1\int_\Omega\left(u_1\frac{\partial\sigma_{12}}{\partial x_1}+u_2\frac{\partial\sigma_{12}}{\partial x_2}\right)\tau_{12}+\int_\Omega\sigma_{12}\tau_{12}=\mu_e\int_\Omega\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\tau_{12}\\ \qquad+\lambda_1\int_\Omega\left(\frac{\partial u_2}{\partial x_1}\sigma_{11}+\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\sigma_{12}+\frac{\partial u_1}{\partial x_2}\sigma_{22}\right)\tau_{12},\\ \lambda_1\int_\Omega\left(u_1\frac{\partial\sigma_{22}}{\partial x_1}+u_2\frac{\partial\sigma_{22}}{\partial x_2}\right)\tau_{22}+\int_\Omega\sigma_{22}\tau_{22}=2\mu_e\int_\Omega\frac{\partial u_2}{\partial x_2}\tau_{22}+2\lambda_1\int_\Omega\left(\frac{\partial u_2}{\partial x_1}\sigma_{12}+\frac{\partial u_2}{\partial x_2}\sigma_{22}\right)\tau_{22}\end{cases} \quad (20)$$

Considering $\lambda_1=1,\mu_e=1$ and writing $\sigma_{11}=d_1,\sigma_{12}=d_2,\sigma_{22}=d_3,\tau_{11}=s_1,\tau_{12}=s_2$ and $\tau_{22}=s_3$ we develop the following code for the solution of the above system:

Here we use the solution of the Navier-Stokes problem. We assume that the velocity is known. We can find the component of stress tensor using the code given below:

```
int n=50, m=50;
real x0=0.0, x1=1.0;
 real y0=0.0, y1=1.0;
 mesh Th=square(n,m,[x0+(x1-x0)*x,y0+(y1-y0)*y]);
plot(Th);
fespace Qh(Th,P1);
Qh d1,d2,d3,s1,s2,s3,d1aux,d2aux,d3aux;
Qh u1=(x^2-x)^2*(y^2-y)*(2*y-1);
Qh u2=-(x^2-x)*(y^2-y)^2*(2*x-1);
real lamda=1.;//error;
real mu=1.;
real eps=10e-8,error;
int i;
problem E1(d1,s1) =int2d(Th)((lamda*(u1*dx(d1)+u2*dy(d1)-2*dx(u1)*d1)+d1)*s1)-
int2d(Th)((2*mu*dx(u1)+2*lamda*dy(u1)*d2aux)*s1);
problem E2(d2,s2) =int2d(Th)((lamda*(u1*dx(d2)+u2*dy(d2)-(dx(u1)+dy(u2))*d2)+d2)*s2)-
int2d(Th)(( mu*(dx(u2)+dy(u1))+lamda*(dx(u2)*d1aux+dy(u1)*d3aux))*s2);
problem E3(d3,s3) =int2d(Th)((lamda*(u1*dx(d3)+u2*dy(d3)-2*dy(u2)*d3)+d3)*s3)-
int2d(Th)((2*mu*dy(u2)+2*lamda*dx(u2)*d2aux)*s3);
d1aux=0; d2aux=0; d3aux=0; error=1; for (i=0;i<=50; i++)
  while (error>eps) { E1; E2; E3;error=int2d(Th)((d1-d1aux)^2+(d2-d2aux)^2+(d3-d3aux)^2);
d1aux=d1;
d2aux=d2;
d3aux=d3;
};
plot(d1);
plot(d2);
plot(d3);
```

Code 2: For Transport Problem

NUMERICAL RESULTS:

We performed the code to find the stress tensor defined on mesh of dimension 50×50. The calculations have been performed with $\lambda_1 = 1$ and $\mu_e = 1$. The unknown stress components are σ_{11}, $\sigma_{12} = \sigma_{21}$ and σ_{22}.

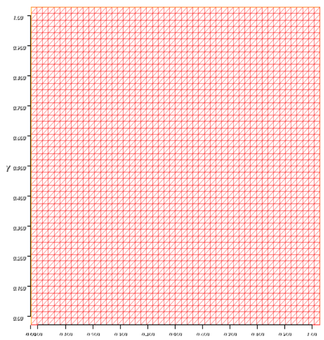

Figure5: 50×50 mesh

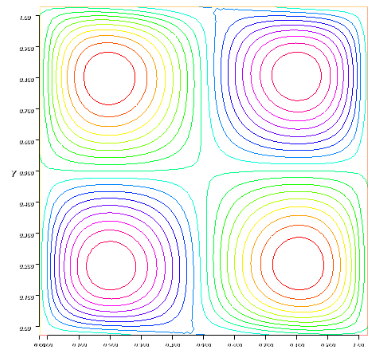

Figure 6: Contours of σ_{11}

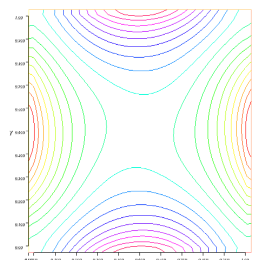

Figure 7: Contours of $\sigma_{12} = \sigma_{21}$

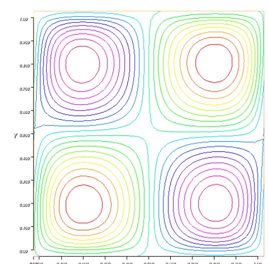

Figure 8: Contours of σ_{22}

We observe that we have the contour plots of the components of the extra stress which are in fact the plot of the solutions of the transport problem. So using FreeFem++ we can obtain the solution of Transport problem.

DISCUSSION AND CONCLUSIONS:

Since Oldroyd-B model can be decoupled into two auxiliary problems namely the Navier-stokes problem and transport problem, so we can find the velocity and pressure i.e. the solution of Navier-Stokes equations from Code 1 using FreeFem++. Then by using this known velocity we find the components of visco-elastic stress tensor i.e. the solution of the Transport problem from code 2. Therefore we obtain, in fact, the solution of the constitutive equation of Oldroyd-B fluid. We can find the velocity, pressure and extra stress tensor for any other problems by using their external forces. Here we approximate the extra stress, velocity and pressure via P_1 continuous, P_2 continuous and P_1 continuous finite element respectively. If the approximation of visco-elastic tensor is P_1 discontinuous finite element, then we can try to find the solution of the Oldroyd-B constitutive equations using FreeFem++ for further work.

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