BAYESIAN ESTIMATION OF THE PARAMETER OF GENERALIZED EXPONENTIAL DISTRIBUTION USING MARKOV CHAIN MONTE CARLO METHOD IN OPEN BUGS FOR INFORMATIVE SET OF PRIORS

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ABSTRACT

The two-parameter generalized exponential (GE) distribution was introduced and studied quite extensively by (Gupta and Kundu (1999), (2001a) and (2001b)). and Ahsanullah (2001) and Raqab (2002) studied the properties of order and record statistics from the GE distribution and their inferences, respectively. Gupta and Kundu (2003) used the ratio of the maximized likelihoods (RML) in discriminating between the Weibull and GE distributions. The two parameters of a GE distribution represent the scale and the shape parameters and because of the scale and shape parameters.

The Markov Chain Monte Carlo (MCMC) method is used to estimate the parameters of a generalized exponential distribution based on a complete sample. The MCMC methods have been shown to be easy to implement computationally, the estimates always exist and are statistically consistent, and their probability intervals are convenient to construct. The R functions are developed to study the statistical properties of the distribution and the output analysis of MCMC samples generated from Open BUGS. The maximum-likelihood estimation (MLE) is the most used method for parameter estimation. We also compute the maximum likelihood estimate and associated confidence intervals to compare the performance of the Bayes estimators with the classical estimator's construction of associated probability intervals. We also develop a module dgen (alpha,lamda) which is written in component Pascal, enable to perform full Bayesian analysis of generalized exponential distribution. The proposed methodology can be applied for empirical modeling, which includes estimation of parameters, model validation and comparison. A real data set is considered for illustration purpose under gamma priors.

Key Words: Generalized Exponential (GE) Distribution, parameter estimation, informative set of priors, Maximum likelihood Estimation (MLE), Bayesian Estimation in Open Bugs, Markov Chain Monte Carlo (MCMC) Method, etc.

1 INTRODUCTION:

Exponential distribution play a central role in analyses of lifetime or survival data, in part because of their convenient statistical theory, their important 'lack of memory' property and their constant hazard rates. In circumstances where the one-parameter family of exponential distributions is not sufficiently broad, a number of wider families such as the gamma, Weibull and lognormal distributions are in common use. Adding parameters to a well-established family of distributions is a time honored device for obtaining more flexible new families of distributions

The probability density function of GE is written as;

$$f(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left\{ 1 - e^{-\lambda x} \right\}^{\alpha - 1} \quad ; (\alpha, \lambda) > 0, x > 0 \qquad (1.1)$$

The typical generalized exponential density functions for different values of   and for   = 1 shows that the density function of the generalized exponential distribution can take different shapes. It is a unimodal density function and for fixed scale parameter as the shape parameter increases it is becoming more and more symmetric. It is unimodal for   > 1 and it is reversed `J' shaped for   1. The density function of the generalized exponential distribution is log-convex if   < 1 and log-concave if   1.

In this paper, the Markov chain Monte Carlo (MCMC) method is used to estimate the parameters of a generalized exponential distribution based on a complete sample. A procedure is developed to estimate the scale and shape parameters of the generalized exponential distribution using Markov Chain Monte Carlo (MCMC) simulation method in Open BUGS. A module (code) is incorporated in Open BUGS. The MCMC methods have been shown to be easy to implement computationally, the estimates always exist and are statistically consistent, and their probability intervals are convenient to construct. The R functions are developed to study the statistical properties of the distribution and the output analysis of MCMC samples generated from Open BUGS. One real data set has been analyzed for illustration.

The rest of the paper is organized as follow:

In section 2, we carry out the model analysis by discussing various function of GE distribution. In section 3; we discuss the MLE and information matrix. We give a brief description of model validation by taking a real data set in section 4. The Bayesian estimation in Open BUGS and its implementation under informative set of priors which include convergence diagnostic, visual summary and numerical summary is discussed in section 5. In section 6, we have made comparison with MLE by way of plotting different graph. The conclusion is given in section 7.

2. Model Analysis

The exponential distribution is generated by a method of introducing a parameter to expand the family of one-parameter exponential distribution.

Let the reliability/survival function is given by

$$R(x; \alpha, \lambda) = 1 - \left\{ 1 - e^{-\lambda x} \right\}^{\alpha} \quad ; (\alpha, \lambda) > 0, \quad x > 0 \qquad (2.1)$$

The R function sgen.exp ( ) computes the reliability/ survival function.

The cumulative distribution function of generalized exponential (GE) distribution

$$F(x; \alpha, \lambda) = \left\{ 1 - e^{-\lambda x} \right\}^{\alpha} \quad ; (\alpha, \lambda) > 0, \quad x > 0 \quad ; \qquad (2.2)$$

where   0 and   0 are the shape and scale parameters, respectively.

The probability density function (pdf) associated with (2.2) is given by

$$f(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left\{ 1 - e^{-\lambda x} \right\}^{\alpha - 1} \quad ; (\alpha, \lambda) > 0, x > 0$$

$$(2.3)$$

We denote the generalized exponential with parameters   and   as X~GE ( )

The R functions dgen.exp ( ) and pgen.exp ( ) can be used for the computation of pdf and cdf, respectively. Some of the typical generalized exponential density functions for different values of   and for   = 1 are shown in Figure 1.
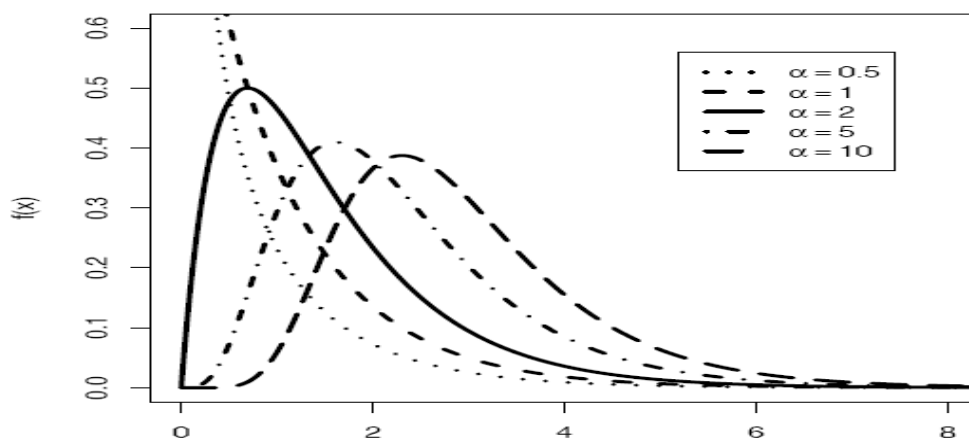
Figure 1 : Plots of the probability density function for =1 and different values of   It is clear from the Figure 1 that the density function of the generalized exponential   distribution can take different shapes. The GE (..) distribution is unimodal with mode at

$$\text{Mode} \quad = \quad \frac{1}{\lambda} \log \alpha \quad ; \alpha > 1 \tag{2.4}$$

and it is reverse 'J' shaped if   < 1. The mean, median, variance and mode all are non-linear functions of the shape parameter, and as the shape parameter goes to infinity, all of them tend to infinity.

$$\text{Mean} \quad = \frac{1}{\lambda} \left\{ \psi(\alpha + 1) - \psi(1) \right\} ; \tag{2.5}$$

$$\text{Variance} = \quad \frac{1}{\lambda^2} \left\{ \psi'(1) - \psi'(\alpha + 1) \right\} \tag{2.6}$$

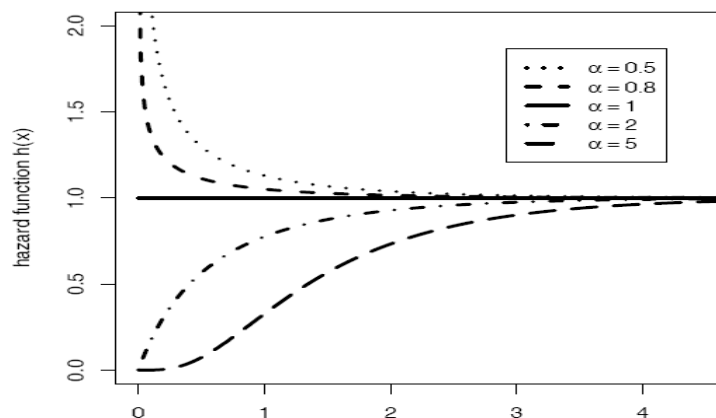$$\text{Median} \quad = \quad -\frac{1}{\lambda} \log \left\{ 1 - (0.5)^{1/\alpha} \right\}. \tag{2.7}$$

where   (.) is the digamma function and   '(.) is its derivative. For large values of   , the mean, median and mode all are approximately equal to log ~

THE HAZARD FUNCTION:

The hazard function is defined as

$$h(x; \alpha, \lambda) = \frac{\alpha \lambda e^{-\lambda x} \left\{ 1 - e^{-\lambda x} \right\}^{\alpha - 1}}{1 - \left\{ 1 - e^{-\lambda x} \right\}^{\alpha}}, \tag{2.8}$$

and the associated R function is hgen.exp( ).Some of the typical hazard functions for different values of ~and for λ=1 are depicted in   Figure 2.

Figure 2   **:** Plots of the hazard function for different values of   and   =

The cumulative hazard function H(x) defined as

$$H(x) = -\{1 - \log F(x)\} \tag{2.9}$$

It can be obtained with the help of pgen.exp( ) function by choosing arguments lower.tail=FALSE and log.p=TRUE. i.e.

pgen.exp(x, alpha, lambda, lower.tail=FALSE,log.p=TRUE)

The survival function (s.f.) and the conditional survival of X are defined by

$$R(x) = 1 - F(x)$$

$$\text{and} \qquad R(x \mid t) = \frac{R(x + t)}{R(x)} \quad , t > 0, x > 0, R(\cdot) > 0, \tag{2.10}$$

respectively, where F(·) is the cdf of X. Similarly to h(x) and FRA(x)

The R functions hra.gen.exp ( ) and crf.gen.exp ( ) are used for the computation of failure rate average (fra) and conditional survival function (crf), respectively.
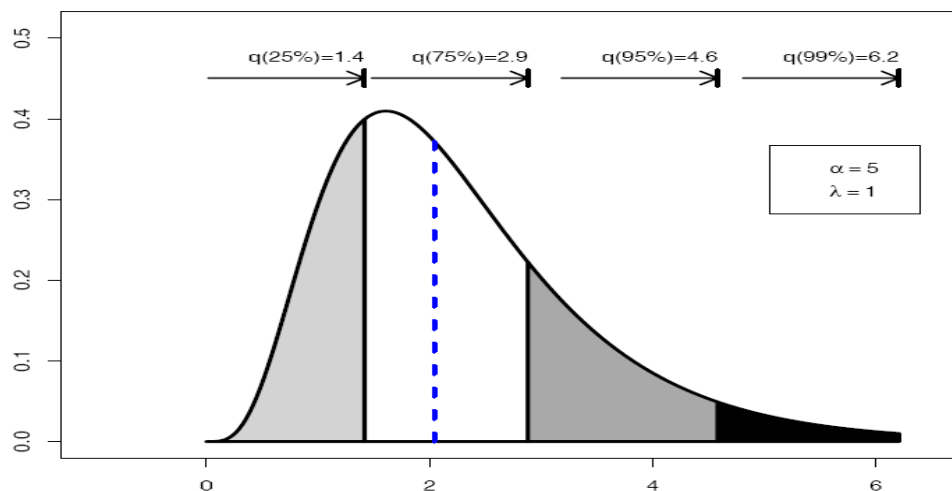
Graphically, the five numbers are often displayed as a box plot.

The quantile function is given by

$$x_q = -\frac{1}{\lambda} \log\left\{1 - q^{1/\alpha}\right\} \quad ; 0 < q < 1. \tag{2.11}$$

The computation of quantiles, the R function qgen.exp ( ) can be used.

Figure 3 illustrates the use of qgen.exp ( ) function.

Figure 3 : An illustration of the quantiles for =1 and vertical line Indicates the median.

3. MAXIMUM LIKELIHOOD ESTIMATION (MLE) AND INFORMATION MATRIX:

Let $\underline{x}=(x_1, \ldots , x_n)$ be a random sample of size n from GE(α, λ), then the log-likelihood function L(α, λ) can be written as;

$$L(\alpha, \lambda) = n \log \alpha + n \log \lambda - \lambda \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} \log\left\{1 - e^{-\lambda x_i}\right\} \tag{3.1}$$

Therefore, to obtain the MLE's of α and λ we can maximize (3.1) directly with respect to α and λ or we can solve the following two non-linear equations using iterative procedure:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left\{1 - e^{-\lambda x_i}\right\} = 0 \tag{3.2}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{\left\{1 - e^{-\lambda x_i}\right\}} = 0 \tag{3.3}$$

Asymptotic Confidence intervals based on MLE Since the MLEs of the unknown parameters =(,) can not be in closed forms, it is not easy to derive the exact distributions of the MLEs. We can derive the asymptotic confidence intervals of these parameters when > 0, and > 0. The large sample approach is to assume that

the MLE $(\hat{\alpha}, \hat{\lambda})$ are approximately bivariate normal with mean (,) and covariance matrix I_0^{-1}, (Lawless (2003)), where I_0^{-1} is the inverse of the observed information matrix

$$
I_0^{-1} = \begin{pmatrix} -\dfrac{\partial^2 \ln L}{\partial \alpha^2}\bigg|_{\hat{\alpha}, \hat{\lambda}} & -\dfrac{\partial^2 \ln L}{\partial \alpha\, \partial \lambda}\bigg|_{\hat{\alpha}, \hat{\lambda}} \\ -\dfrac{\partial^2 \ln L}{\partial \lambda\, \partial \alpha}\bigg|_{\hat{\alpha}, \hat{\lambda}} & -\dfrac{\partial^2 \ln L}{\partial \lambda^2}\bigg|_{\hat{\alpha}, \hat{\lambda}} \end{pmatrix}^{-1} = \left(-H\big|_{(\hat{\alpha}, \hat{\lambda})}\right)^{-1}
$$

$$
= \begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) & \operatorname{var}(\hat{\lambda}) \end{pmatrix} \qquad (3.4)
$$

The above approach is used to derive the $100(1 - /2)\%$ confidence intervals of the parameters $=($, $)$ as in the following forms

$$
\hat{\alpha} \pm z_{\gamma/2}\sqrt{\operatorname{Var}(\hat{\alpha})} \quad \text{and} \quad \hat{\lambda} \pm z_{\gamma/2}\sqrt{\operatorname{Var}(\hat{\lambda})} \qquad (3.5)
$$

Here, $z_{/2}$ is the upper $(/2)^{th}$ percentile of the standard normal distribution.

COMPUTATION OF MAXIMUM LIKELIHOOD (ML) ESTIMATION:

We consider the real data set reported in (Lawless, 2003, page 228) presented here arose in test on endurance of deep groove ball bearing. The data represented are the number of million revolutions before failure of each of the 23 ball bearing in the life test. The direct maximization of log-likelihood function given in (3.1) using Newton-Raphson method in R gives, the ML estimates and standard error. The 95% confidence interval is computed using (3.4) and (3.5).
The Table 1. Shows the ML estimates, standard error (SE) and 95 % Confidence Intervals for parameters alpha and lambda

TABLE 1 : ML ESTIMATE, STANDARD ERROR AND 95% CONFIDENCE INTERVAL

Parameter	MLE	Std. Error	95% Confidence Interval
alpha	5.28321	2.04816	(1.26881, 9.29761)
lambda	0.03230	0.00642	(0.01971, 0.04488)

4. MODEL VALIDATION:

Most statistical methods assume an underlying distribution (model) in the derivation of their results. However, when we assume that the data follow a specific distribution, we are making an assumption. If such a model does not hold, then the conclusions from such analysis may be invalid. For example, the confidence levels of the confidence intervals or hypotheses tests implemented may be off. Although hazard plotting and the other graphical methods can guide the choice of the parametric distribution, one cannot of course be sure that the proper model has been selected. Hence model verification is still necessary to check whether we have achieved the goal of choosing the right model. In this section we outline some of the methods used to check model appropriateness.
There are two graphical methods widely used for checking whether a fitted model is in agreement with the data. In other words, we measure the quality of the fitted model. These methods are Quantile-Quantile (Q-Q) plots and the Probability-Probability (P-P) plots in model validation.

THE Q-Q PLOTS

Let $\hat{F}(x)$ be an estimate of $F(x)$ based on $x_1, x_2, \ldots, x_n$. The scatter plot of the points

$$\hat{F}^{-1}(p_{1:n}) \quad \text{versus} \quad x_{i:n}, \quad i = 1, 2, \ldots, n, \text{ is called a Q-Q plot.}$$

Thus, the Q-Q plots show the estimated versus the observed quantiles. If the model fits the data well, the pattern of points on the Q-Q plot will exhibit a 45-degree straight line. Note that all the points of a Q-Q plot are inside the square

$$\left[\hat{F}^{-1}(p_{1:n}),\ \hat{F}^{-1}(p_{n:n})\right]\ \times\ \left[x_{1:n},\ x_{n:n}\right].$$

The corresponding R function is qq.gen.exp ( ). It can be seen from the straight line pattern in Figure 4; the GE fits the data very well.
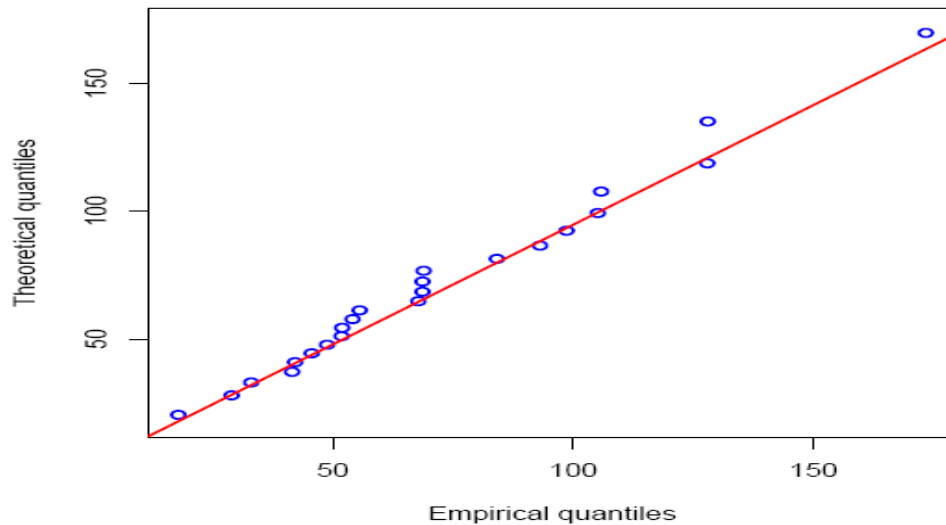

Figure 4 : Quantile-Quantile (Q-Q) plot using MLEs as estimate.

THE P-P PLOT:

Let $x_1, x_2, \ldots, x_n$ be a sample from a given population with estimated cdf $\hat{F}(x)$. The scatter plot of the points

$$\hat{F}(x_{1:n})\quad \text{versus}\quad p_{i:n},\ i=1,2,\ldots,n,$$

is called a P-P plot. If the model fits the data well, the graph will be close to the 45-degree line. We see that all the points in the P-P plot are inside the unit square [0, l] x [0, 1]. The corresponding R function is pp.gen.exp ( ).It can be seen from the straight line pattern in Figure 5 the GE distribution fits the data very well.
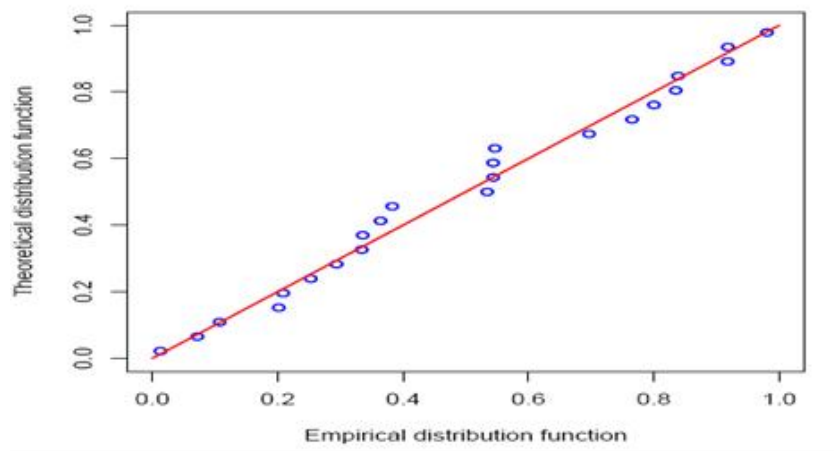
Figure 5 : Probability-Probability (P-P) plot using MLEs as estimate

5. Bayesian Estimation in Open BUGS:

Statistical models must be described before they can be used. A language to describe statistical models is needed by both the users of the model and the software that makes inference about the model. The language should be a formal language with well defined rules which can be processed automatically as for as possible to make inferences about the model. A small change to a model should not lead to a large change in the way that model is described

The most widely used piece of software for applied Bayesian inference is the WinBUGS package, distributed by the MRC Biostatistics Unit at Cambridge (Spiegelhalter, Thomas, Best, and Lunn 2004).

A module dgen.exp (alpha, lambda) is written in component Pascal to perform full Bayesian analysis of generalized exponential (GE) model into Open BUGS using the method described in Thomas (2007).

IMPLEMENTATION OF MODULE - DGEN.EXP (ALPHA, LAMBDA) :

The developed module is implemented to obtain the Bayes estimates of the GE distribution using MCMC method. The main function of the module is to generate MCMC sample from posterior distribution for given set of priors.

Gamma prior:

The most widely used prior distribution of is the inverted gamma distribution with Parameters a and b (>0) with p.d.f. given by;

$$\pi(\theta) = \begin{cases} \dfrac{b^a}{\Gamma(a)} \, \theta^{a-1} \quad e^{-b\,\theta} & ; \theta > 0 \quad (a,b) > 0 \\ 0 & ; otherwise \end{cases}$$

Bayesian Analysis under Informative set of Priors (Gamma)

Model
```
{
        for( i in 1 : N )
          {
              x[i] ~ dgen.exp(alpha, lambda)

          }
# Prior distributions of the Model parameters
# Gamma prior for alpha
alpha~ dgamma(0.001, 0.001)
# Gamma prior for lambda
lambda~ dgamma(0.001, 0.001)
          }
```

DATA SET:

list(N=23, x = c(17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40))

INITIAL VALUES :

```
    # Chain1
        list(alpha=4.0, lambda=0.01)

    # Chain 2
    list(alpha=8.0, lambda=0.1)
```

We run the model to generate two Markov Chains at the length of 40,000 with different starting points of the parameters. The convergence is monitored using trace and ergodic mean plots, we find that the Markov Chain converge together after approximately 2000 observations. Therefore, burning of 5000 samples is more than enough to erase the effect of starting point (initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome, i.e. thin=5, starting from 5001.This is done to minimize the auto correlation among the generated deviates.

Therefore, we have the posterior sample $\{\alpha_{1i}, \lambda_{1i}\}$, i = 1,…,7000 from chain 1 and $\{\alpha_{2i}, \lambda_{2i}\}$, i = 1,…,7000 from chain 2.

The chain 1 is considered for convergence diagnostics plots. The visual summary is based on posterior sample obtained from chain 2 whereas the numerical summary is presented for both the chains.

CONVERGENCE DIAGNOSTICS:

History (Trace) plot Before examining the parameter estimates or performing other inference, it is a good idea to look at plots of the sequential (dependent) realizations of the parameter estimates and plots thereof.

We have found that if the Markov chain is not mixing well or is not sampling from the stationary distribution, this is usually apparent in sequential plots of one or more realizations. The sequential plot of parameters is the plot that most often exhibits difficulties in the Markov chain. Figure 6 shows the sequential realizations of the parameters of the model. In this case Markov chain seems to be mixing well enough and is likely to be sampling from the stationary distribution.
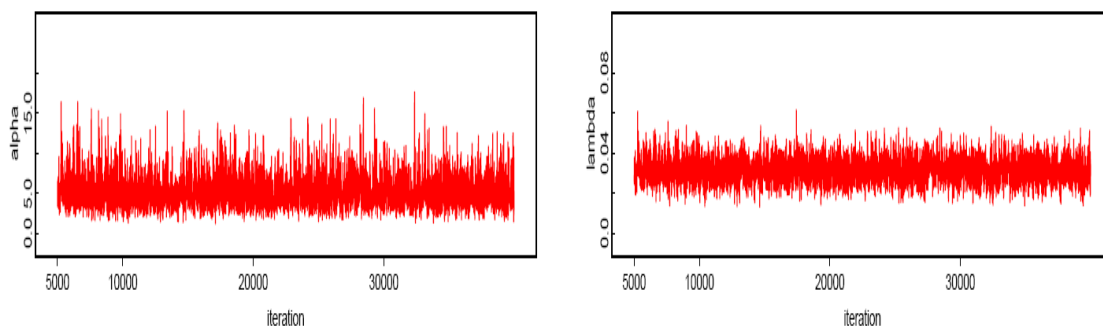

Figure 6 : Sequential realization of the parameters and .

The plot looks like a horizontal band, with no long upward or downward trends, so we have evidence that the chain has converged.

RUNNING MEAN (ERGODIC MEAN) PLOT:

Generate a time series (iteration number) plot of the running mean for each parameter in the chain. The running mean is computed as the mean of all sampled values up to and including that at a given iteration.
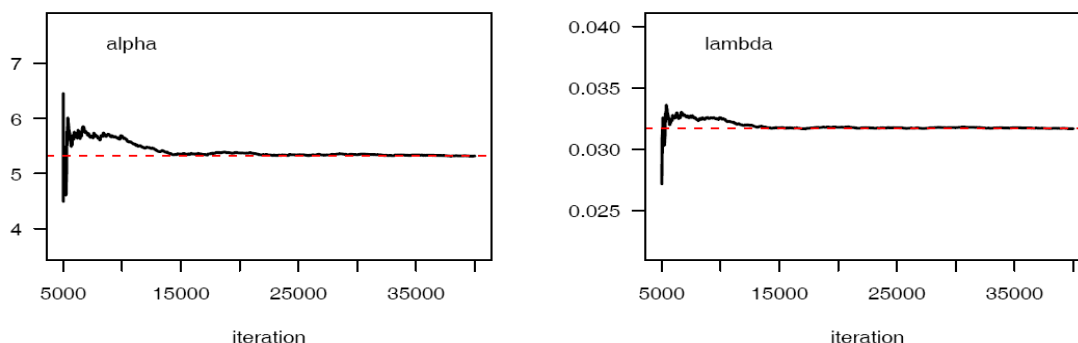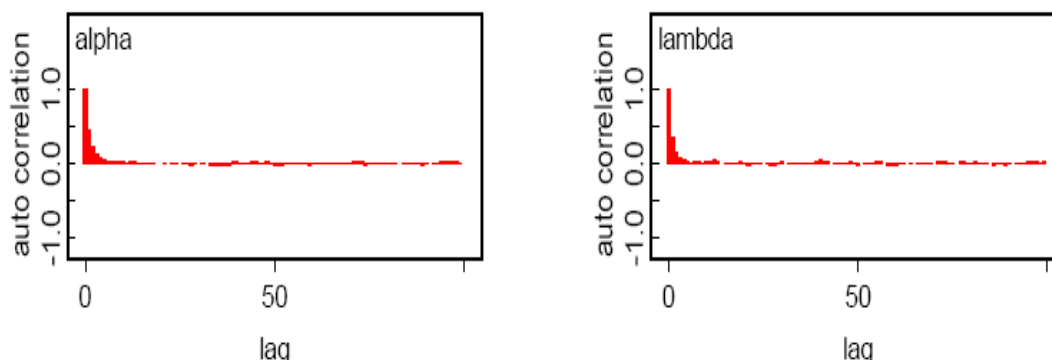

Figure 7 : The Ergodic mean plots for alpha and lambda.

The convergence pattern based on ergodic averages is shown in Figure 7 which indicates the convergence of the chain

AUTOCORRELATION:

The graph shows that the correlation is almost negligible. We may conclude that the samples are independent.

Figure 8 : The autocorrelation plots for alpha and lambda

NUMERICAL SUMMARY:

In Table 2. We have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for generalized exponential (GE) distribution under gamma priors. The Highest probability density (HPD) intervals are computed the algorithm described by Chen and Shao (1999) under the assumption of unimodal marginal posterior distribution.

TABLE 2: MCMC SAMPLE BASED POSTERIOR CHARACTERISTICS FOR GENERALIZED EXPONENTIAL DISTRIBUTION UNDER GAMMA PRIORS

| Characteristics	Chain 1		Chain 2	
	alpha	lambda	alpha	lambda
Mean	5.319	0.03169	5.273	0.03155
Standard Deviation	2.168	0.006492	2.139	0.006389
Monte Carlo(MC) error	0.04123	0.000117	0.04336	0.000124
Minimum	1.188	0.01323	1.032	0.01116
First Quartile (Q_1)	3.755	0.02715	3.74	0.02714
Median	4.943	0.03164	4.886	0.03131
Third Quartile (Q_3)	6.433	0.03587	6.344	0.03575
Maximum	17.61	0.06143	19.26	0.06784
Mode	4.37860	0.03272	4.36888	0.03075
2.5th Percentile($P_{2.5}$)	2.232	0.01968	2.225	0.0201
97.5th Percentile($P_{97.5}$)	10.66	0.04503	10.5	0.04489
95% Credible Interval	(2.232, 10.66)	(0.01968, 0.04503)	(2.225, 10.5)	(0.0201, 0.04489)
95% HPD Credible Interval	(1.782, 9.676)	(0.01960, 0.04489)	(1.873, 9.508)	(0.01977, 0.04425)

VISUAL SUMMARY:

Box plots:

The boxes represent inter-quartile ranges and the solid black line at the (approximate) centre of each box is th mean; the arms of each box extend to cover the central 95 percent of the distribution - their ends correspond, therefore, to the 2.5% and 97.5% quartiles.
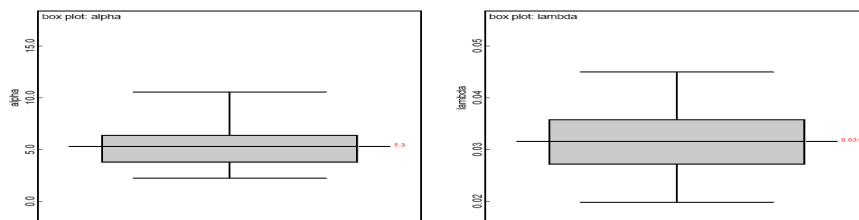
**Figure 9 :** The box plots for alpha and lambda.

6.   COMPARISON WITH MLE:

For the comparison with MLE we have plotted two graphs. In Figure 10 the density functions $f(x; \hat{\alpha}, \hat{\lambda})$ using MLEs and Bayesian estimates, computed via MCMC samples under gamma priors, are plotted.
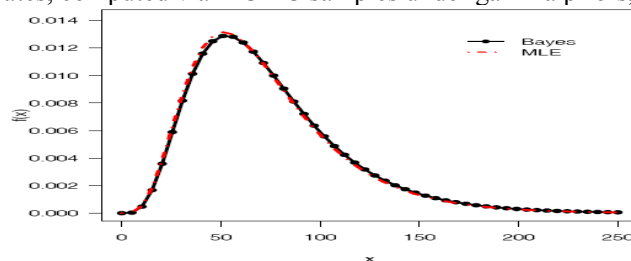
Figure 10 : The density functions $f(x; \hat{\alpha}, \hat{\lambda})$ using MLEs and Bayesian estimates,Computed via MCMC samples under gamma priors.

The Figure 11 represent the Quantile-Quantile(QQ) plot of empirical quantiles and theoretical quantiles computed from MLE and Bayes estimates
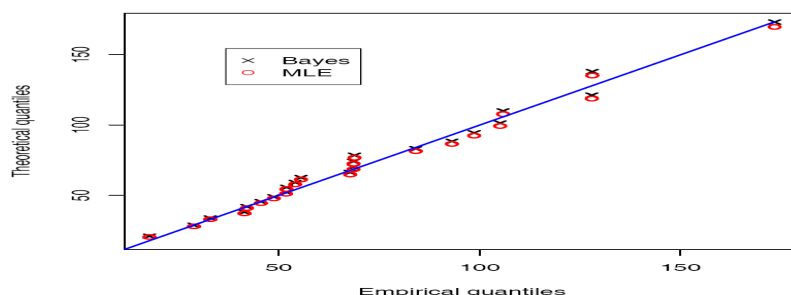
Figure 11 : Quantile-Quantile (QQ) plot of empirical quantiles and theoretical quantiles computed from MLE and Bayes estimates.

It is clear from the Figures 11; the MLEs and the Bayes estimates with respect to the gamma priors are quite close and fit the data very well.

7.   CONCLUSION:

The developed methodology for MLE and Bayesian estimation has been demonstrated on a real data set when both the parameters alpha (shape) and lambda (scale) of the GE distribution are unknown under informative set of independent priors. The bayes estimates of the said priors have been obtained under squared error, absolute error and zero-one loss functions. A five point summary Minimum (x), Q_1, Q_2, Q_3, Maximum (x) has been computed. The symmetric Bayesian credible intervals and Highest Probability Density (HPD) intervals have been constructed. With the use of graphical representations the intent is that one can gain a perspective of various meanings and associated interpretations.

The MCMC method provides an alternative method for parameter estimation of the GE distributions. It is more flexible when compared with the traditional methods such as MLE method. Moreover, 'exact' probability intervals are available rather than relying on estimates of the asymptotic variances. Indeed, the MCMC sample may be used to completely summarize posterior distribution about the parameters, through a kernel estimate. This is also true for any function of the parameters such as hazard function, mean time to failure etc. The MCMC procedure can easily be applied to complex Bayesian modeling relating to GE model.

REFERENCES:

[1]   Albert, J. (2007). Bayesian Computation with R, Springer, New York.
[2]   Berger, J. O. (1985). Statistical Decision Theory and Bayesian Analysis, Second Edition, Springer-Verlag, New York.
[3]   Bernardo, J. M. and Smith, A. F. M. (2000). Bayesian Theory, Wiley.
[4]   Bhattacharya, S. K. (1967). Bayesian approach to life testing and reliability, Journal of the American Statistical Association, 62, 48-62.
[5]   Braun, W. J. and Murdoch, D. J. (2007). A First Course in Statistical Programming with R, Cambridge University Press, Cambridge.
[6]   Brooks, S. and Roberts, G. (1999). On Quantile Estimation and Markov Chain Monte Carlo Convergence, Biometrika, 86(3), 710–717.
[7]   Chen, M., Shao, Q. and Ibrahim, J.G. (2000). Monte Carlo Methods in Bayesian Computation, Springer, NewYork.
[8]   Chib S. and Greenberg E. (1995). Understanding the Metropolis Algorithm, The American Statistician, 49, 327-335.
[9]   Congdon, P. (2003). Applied Bayesian Modelling, John Wiley.
[10] Congdon, P. (2006). Bayesian Statistical Modelling, 2nd edition. John Wiley.
[11] Gamerman, D. (1997). Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference, London, Chapman and Hall.
[12] Gelfand, A. E., Hills, S. E., Racine-Poon, A., and Smith, A. F. M. (1990). Illustration of Bayesian Inference in Normal Data Models Using Gibbs Sampling, Journal of the American Statistical Association, 85, 972–985.
[13] Ghosh, J.K., Delampady, M. and Samanta, J.(2006). An Introduction to Bayesian Analysis: Theory and Methods, Springer Science, New York.

[14] Gilks, W.R., Best, N.G., Tan, K.K.C. (1995). Adaptive rejection metropolis sampling. Applied Statistics, 44 (4), 455–472.

[15] Gilks, W.R., Richardson, S., Spiegelhalter, D.J. (1996). Markov chain Monte Carlo in Practice, Chapman and Hall, London.

[16] Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions, Australian and New Zealand Journal of Statistics, 41(2), 173 - 188.

[17] Gupta, R. D. and Kundu, D. (2001a). Generalized exponential distributions: different methods of estimation, Journal of Statistical Computation and Simulation. 69, 315 - 338.

[18] Gupta, R. D. and Kundu, D. (2001b). Exponentiated exponential family; an alternative to gamma and Weibull distributions, Biometrical Journal, 43(1), 117 - 130.

[19] Gupta, R. D. and Kundu, D. (2002). Generalized exponential distributions: statistical inferences, Journal of Statistical Theory and Applications, 1, 101 - 118.

[20] Gupta, R.D. and Kundu, D. (2001). Exponentiated exponential distributions, different methods of estimations. Journal of Statistical Computation and Simulations, **69**(4), 315–338.

[21] Gupta, R.D. and Kundu, D. (2003). Discriminating between Weibull and generalized exponential distributions. Computational Statistics and Data Analysis, 43, 179–196.

[22] Gupta, R.D. and Kundu, D. (2007). Generalized exponential distribution: existing results and some recent developments, Journal of Statistical Planning and Inference, doi: 10.1016/j.jspi.2007.03.030.

[23] Ibrahim, J. G., Chen, M. H. and Sinha, D.(2001). Bayesian Survival Analysis, Springer, New York.

[24] Jaheen, Z. F. (2004). Empirical Bayes inference for generalized exponential distribution based on records, Communications in Statistics- Theory and Methods, 33(8), 1851-1861.

[25] Kundu, D. and Gupta, R.D. (2007). A convenient way of generating gamma random variables using generalized exponential distribution, Computational Statistics and Data Analysis, 51, 2796-2802.

[26] Kundu, D. and Gupta, R.D. (2008). Generalized Exponential Distribution. : Bayesian Estimations, Computational Statistics and Data Analysis, 52(4), 1873-1883.

[27] Lunn, D.J., Andrew, A., Best, N. and Spiegelhalter, D. (2000). WinBUGS – A Bayesian modeling framework: Concepts, structure, and extensibility, Statistics and Computing, 10**,** 325–337

[28] Metropolis, N. and Ulam, S. (1949). The Monte Carlo Method, Journal of the American Statistical Association, 44, 335–341.

[29] Nelson, W. (1982). Applied Lifetime Data Analysis, Wiley, New York.

[30] Raftery, A. E. and Lewis, S. M. (1992). One Long Run with Diagnostics: Implementation Strategies for Markov Chain Monte Carlo, Statistical Science, 7, 493–497.

[31] Raqab, M.Z. and Madi, M.T. (2005). Bayesian inference for the generalized exponential distribution, Journal of Statistical Computation and Simulation, 75(10), 841-852.

[32] Sinha, S.K. (1998). Bayesian Estimation, New Age International, New Delhi.

[33] Spiegelhalter, D., Thomas, A., Best, N., Lunn, D. (2004). WinBUGS, Version 1.4.1, URL http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/.

[34] Wasserman, L. (2004). All of Statistics: A Concise Course in Statistical Inference, New York, Springer-Verlag.

[35] Weibull, W. (1951). A statistical distribution functions of wide applicability, Jour. Appl. Mech., 18, 293-297.

[36] Wingo, D.R. (1983). Maximum likelihood methods for fitting the Burr Type XII distribution parameters to life test data, Biometrical Journal, 25, 77–84.

[37] Wingo, D.R. (1993). Maximum likelihood estimation of Burr XII distribution parameters under Type II censoring, Microelectronics and Reliability, 33, 1251–1257.

[38] Wu Jong-Wuu and Yu Huei-Yi(2005). Statistical inference about the shape parameter of the Burr type XII distribution under the failure-censored sampling plan, Applied Mathematics and Computation, 163(1), 443-482.

[39] Zacks, S. (1981). Parametric Statistical Inference: Basic Theory and Modern Approaches, Pergamon Press, Oxford.

[40] Zacks, S. (1992). Introduction to Reliability Analysis, Probability Models and Statistical Methods, Springer-Verlag, Berlin.

[41] Zehna (1966). Invariance of Maximum Likelihood Estimation, Annals of Mathematical Statistics, 66, 891.

[42] Zheng, G. (2002). Fisher Information Matrix in Type-II Censored Data from Exponentiated Exponential Family, Biometrical Journal, 44, 353 – 357.

[43] Zimmer, W. J., Keats, J. B. and Wang, F. K. (1998). The Burr XII distribution in reliability analysis, Journal of Quality Technology, 30(4), 386–394.